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# Global regularity for solutions to Dirichlet problem for discontinuous elliptic systems with nonlinearity $q > 1$ and with natural growth

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**Abstract** In this paper we deal with the Hölder regularity up to the boundary of the solutions to a nonhomogeneous Dirichlet problem for second-order discontinuous elliptic systems with nonlinearity $q > 1$ and with natural growth. The aim of the paper is to clarify that the solutions of the above problem are always global Hölder continuous in the case of the dimension $n = q$ without any kind of regularity assumptions on the coefficients. As a consequence of this sharp result, the singular sets $\Omega_0 \subset \Omega$, $\Sigma_0 \subset \partial\Omega$ are always empty for $n = q$. Moreover we show that also for $1 < q < 2$, but $q$ close enough to 2, the solutions are global Hölder continuous for $n = 2$.

**Keywords** Nonlinear elliptic systems · Global Hölder regularity · Higher gradient summability

**Mathematics Subject Classification (2000)** MSC 35J65 · MSC 35J55

## 1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with boundary $\partial\Omega$ of class $C^2$; let $q$ be a real number $> 1$ and $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, $s > q$. If $u : \Omega \to \mathbb{R}^N$, we set $Du = (D_1 u, \ldots, D_n u)$ and we denote by $p = (p^1, \ldots, p^n)$, with $p^i \in \mathbb{R}^N$, a typical vector of $\mathbb{R}^{nN}$ and by $(u|v)$, with $u, v \in \mathbb{R}^N$, the inner product in $\mathbb{R}^N$.

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The aim of this paper is to study the global Hölder continuity in $\overline{\Omega}$ of a solution $u \in H^{1,q}(\Omega) \cap L^\infty(\Omega)$ to the following Dirichlet problem

$$\begin{cases} u - g \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega) \\ \sum_{i=1}^n D_i a^i(x, u, Du) = -B(x, u, Du) \quad \text{in } \Omega \end{cases} \tag{1}$$

where $a^i(x, u, p)$, $i = 1, 2, \ldots, n$, are vectors of $\mathbb{R}^N$, defined on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$, such that $a^i(x, u, p)$ are measurable in $x$ and continuous in $u$, $p$, and $a^i(x, u, 0) = 0$ for a.a. $x \in \Omega$, $\forall u \in \mathbb{R}^N$.

For solution $u$ to (1) we mean that $u = g + w$, where $w \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ is such that

$$\begin{aligned} &\int_\Omega \sum_{i=1}^n \left(a^i(x, w + g, Dw + Dg)|D_i\varphi\right) \mathrm{d}x \\ &= \int_\Omega (B(x, w + g, Dw + Dg)|\varphi) \, \mathrm{d}x, \qquad \forall \varphi \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega). \end{aligned} \tag{2}$$

We do not require regularity assumptions on the coefficients, but setting $\forall p \in \mathbb{R}^K$, $K \geq 1$

$$V(p) = \left(1 + \|p\|^2\right)^{\frac{1}{2}} \quad \text{and} \quad W(p) = V^{\frac{q-2}{2}}(p)\, p \tag{3}$$

we only assume that there exist two positive constants $M$, $\nu$ such that for a.a. $x \in \Omega$, $\forall u \in \mathbb{R}^N$, $p \in \mathbb{R}^{nN}$, it results

$$\|a^i(x, u, p)\| \leq M\, V^{q-2}(p)\|p\|, \qquad i = 1, \ldots, n \tag{4}$$

$$\sum_{i=1}^n \left(a^i(x, u, p)|p^i\right) \geq \nu\, V^{q-2}(p)\, \|p\|^2. \tag{5}$$

Let us note that an example of operator satisfying conditions (4), (5) is given by the well known one:

$$\sum_i D_i[(1 + \|Du\|^2)^{\frac{q-2}{2}} D_i u]$$

and that in the case $1 < q < 2$ conditions (4), (5) appear as degenerate conditions.

On the free term $B$ we suppose that there exist two positive constants $a$, $b$, such that for a.a. $x \in \Omega$, $\forall u \in \mathbb{R}^N$, $p \in \mathbb{R}^{nN}$ it results

$$\|B(x, u, p)\| \leq a + b\|W(p)\|^2 \tag{6}$$

and, if $u$ is a solution to Problem (1), also the following smallness condition holds

$$2b\, \|u - g\|_{L^\infty(\Omega)} < \nu. \tag{7}$$

Condition (6) is called natural growth condition. Further we would like to point out that the so called smallness condition (7) is necessary, in a certain sense, in order to obtain the regularity result of the solutions, in virtue of the well known counter-examples, as, for example, the one provided by Freshe [10]; however it seems that the optimal smallness condition could be $b\, \|u - g\|_{L^\infty(\Omega)} < \nu$, even if in the literature it is used condition (7).

Taking into account the counter-examples provided in [7,12,17,25] and the general form of the coefficients $a^i(x, u, Du)$ of problem (1), it is well known that it is not possible to expect the global Hölder continuity of solutions for $n > q$. For $n < q$ the desired regularity easily follows from Sobolev imbedding theorems, then our goal remains only in proving the regularity up to the boundary for $n = q$. For what concerns the case $q \geq 2$ we achieve this result by means of the following theorem on higher global integrability of the gradient:

**Theorem 1** *Assume that conditions* (4)–(7) *are fulfilled. Let $\partial\Omega$ be of class $C^2$ and $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, with $s > q$. If $u \in H^{1,q}(\Omega) \cap L^\infty(\Omega)$, $q \geq 2$, is a solution to Dirichlet Problem* (1)*, then there exists a number $r > 1$ such that*

$$u \in H^{1,qr}(\Omega).$$

From Theorem 1 we immediately derive the following corollary.

**Corollary 1** *Under the same assumptions of Theorem* 1*, a solution $u$ to Dirichlet Problem* (1)*, for $q = n$, belongs to $C^{0,\alpha}(\overline{\Omega})$, with $\alpha = 1 - \dfrac{1}{r}$.*

Considering now the case of nonlinearity $1 < q < 2$, we obtain again a result of higher global integrability of the gradient. In fact we prove the following theorem.

**Theorem 2** *Assume that conditions* (4)–(7) *are fulfilled. Let $\partial\Omega$ be of class $C^2$ and $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, with $s > q$. If $u \in H^{1,q}(\Omega) \cap L^\infty(\Omega)$, $1 < q < 2$, is a solution to Dirichlet Problem* (1)*, then there exists a number $r > 1$ such that*

$$u \in H^{1,qr}(\Omega).$$

From the proof of Theorem 2 it follows that the number $r$, for $q$ belonging to an interval $[q_0, 2]$ with $q_0 > 1$, can be chosen independently from $q$ and hence we can reach the second main result of this paper.

**Corollary 2** *Under the same assumptions of Theorem* 2*, a solution $u$ to Dirichlet Problem* (1)*, for $q \in \left(\dfrac{n}{r}, 2\right)$, belongs to $C^{0,\alpha}(\overline{\Omega})$, with $\alpha = 1 - \dfrac{n}{qr}$.*

It is interesting to note that even in the case $1 < q < 2$, for $n = 2$, we obtain the Hölder continuity up to the boundary for $\dfrac{2}{r} < q < 2$.

An essential tool in order to achieve the global higher summability of $Du$ is to get the so called "Caccioppoli type inequality", both in the interior case and near the boundary, and to apply, both in the interior case and near the boundary again, a refined version of the Gehring Lemma due to Zatorska-Goldstein (see [28]), in which the values of the increment of summability is specified. Similar arguments are used in [9] in order to study the homogenization of a Stampacchia type operator on a Carnot group.

In the general case, namely without requirements on the dimension $n$ and with coefficients $a^i$ depending on $x$, $u$, $Du$, the result we can expect if $q > n$ is only the so called "partial Hölder regularity", that is there exists a closed singular set $\Omega_0$ such that $u$ is Hölder continuous in $\Omega \setminus \Omega_0$ and, even if the trace of $u$ on $\partial\Omega$ is smooth, there exists a closed singular set $\Sigma_0$ on $\partial\Omega$ such that $u$ is Hölder continuous up to the boundary except for the points of $\Sigma_0$ (for nonlinearity $q > 2$ see [4]; for nonlinearity $q = 2$ see [1,3,6,13,18,21,22,27]; it is worth mentioning that all these results are obtained under suitable regularity assumptions on the data). Moreover in particular cases it is possible to estimate, always under suitable regularity assumptions on the data, the Hausdorff dimension of both the singular sets $\Omega_0$ and $\Sigma_0$ (see for example [4,18,21,22]).

The general case with $1 < q < 2$ is less studied than the case $q \geq 2$. We mention that differentiability results for $1 < q < 2$ are obtained in [2,24]. Moreover in [23] a homogeneous system with coefficients of the type $a^i(Du)$ is studied and global Hölder continuity results are obtained.

Also we note that in [26] the author obtains the global Hölder continuity up to the boundary for $n = q$ in the case of term $B$ fulfilling a growth of the type $\|p\|^{q-1}$.

It is worth mentioning some unexpected results when the coefficients are of particular type. For example, if the coefficients are of the type $a^i(x, Du)$, it is possible to improve the Hölder regularity results obtaining the Hölder continuity also for $n \leq q + 2$ (see [5,19]).

Moreover we would point out that the behaviour of weak solutions with respect to the Hölder continuity is analogous to the one we meet when we consider elliptic nonvariational systems, namely we obtain global Hölder continuity up to the boundary only for low values of $n$ and partial Hölder continuity in the general case (see [8,20]).

Finally we would like to mention that the results of this paper in the case $q \geq 2$ have been presented at the International Conference "Variational Analysis and Applications" and an abridged version of these results has been published in [15,16].

## 2 Notations and preliminary results

We set

$$B(x^0, \sigma) = \left\{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\right\}$$

and, if $x_n^0 = 0$,

$$B^+(x^0, \sigma) = \left\{x \in B(x^0, \sigma) : x_n > 0\right\},$$

$$\Gamma(x^0, \sigma) = \left\{x \in B(x^0, \sigma) : x_n = 0\right\}.$$

We will simply write $B(\sigma)$, $B^+(\sigma)$, $\Gamma(\sigma)$ and $\Gamma$ instead of $B(0, \sigma)$, $B^+(0, \sigma)$, $\Gamma(0, \sigma)$ and $\Gamma(0, 1)$, respectively.

Moreover if $u \in L^1(\mathcal{B})$ and $\mathcal{B}$ is a measurable set with meas $\mathcal{B} \neq 0$, then

$$u_{\mathcal{B}} = \fint_{\mathcal{B}} u(x)\mathrm{d}x = \frac{1}{\operatorname{meas} \mathcal{B}} \int_{\mathcal{B}} u(x)\mathrm{d}x.$$

In the sequel we need the following refined version of Gehring Lemma (see [11, 14]) due to Zatorska-Goldstein (see [28]).

**Lemma 1** *Let $t \in [q_0, 2n]$ with $q_0 > 1$ fixed. Assume that $U$ and $G$ are non-negative functions on $\Omega$ such that*

$$U \in L^t(\Omega), \quad G \in L^s(\Omega), \quad 1 < t < s$$

*and if, for every $B(x^0, \sigma) \subset B(x^0, 2\sigma) \subset \Omega$, it results*

$$\fint_{B(x^0,\sigma)} U^t \mathrm{d}x \leq \beta \left\{ \left[ \fint_{B(x^0,2\sigma)} U \mathrm{d}x \right]^t + \fint_{B(x^0,2\sigma)} G^t \mathrm{d}x \right\}, \quad \beta > 1$$

*then there exists $\varepsilon_0 > 0$ such that $U \in L^r_{\mathrm{loc}}(\Omega)$, $\forall r \in [t, \min\{t + \varepsilon_0, s\})$ and*

$$\left( \fint_{B(x^0,\sigma)} U^r \mathrm{d}x \right)^{\frac{1}{r}} \leq k \left\{ \left[ \fint_{B(x^0,2\sigma)} U^t \mathrm{d}x \right]^{\frac{1}{t}} + \left[ \fint_{B(x^0,2\sigma)} G^r \mathrm{d}x \right]^{\frac{1}{r}} \right\}.$$

*The constant* $\varepsilon_0$ *is given by*

$$\varepsilon_0 = \frac{q_0 - 1}{\beta\, 2^{3+16n}\, n^{2n}}$$

*and* $k$ *depends on* $\beta$, $q_0$, $n$.

The following estimate will be also useful in the proof of our results.

**Lemma 2** *There exists a positive constant* $c(q) = \max\{2^{q-2}, 1\}$ *such that,* $\forall q \geq 1$ *and* $\forall p, \tilde{p} \in \mathbb{R}^{nN}$, *it results*

$$V^{q-2}(p + \tilde{p})\, \|p + \tilde{p}\| \leq c(q)\, [V^{q-2}(p)\, \|p\| + V^{q-2}(\tilde{p})\, \|\tilde{p}\|]. \tag{8}$$

*Proof* First let us consider the case $q \geq 2$. We have

$$V^{q-2}(p + \tilde{p})\, \|p + \tilde{p}\| = \left(1 + \|p + \tilde{p}\|^2\right)^{\frac{q-2}{2}}\, \|p + \tilde{p}\|.$$

If $\|p\| \leq \|\tilde{p}\|$, then

$$\begin{aligned}\left(1 + \|p + \tilde{p}\|^2\right)^{\frac{q-2}{2}}\, \|p + \tilde{p}\| &\leq 2\left(1 + 4\|\tilde{p}\|^2\right)^{\frac{q-2}{2}}\, \|\tilde{p}\| \\ &\leq 2\, 4^{\frac{q-2}{2}}\left(1 + \|\tilde{p}\|^2\right)^{\frac{q-2}{2}}\, \|\tilde{p}\| \\ &= 2\, 2^{q-2} V^{q-2}(\tilde{p})\, \|\tilde{p}\|.\end{aligned}$$

In a similar way, if $\|\tilde{p}\| \leq \|p\|$, we get

$$\left(1 + \|p + \tilde{p}\|^2\right)^{\frac{q-2}{2}}\, \|p + \tilde{p}\| \leq 2\, 2^{q-2} V^{q-2}(p)\, \|p\|.$$

Then, by summing the previous inequalities, we obtain

$$V^{q-2}(p + \tilde{p})\, \|p + \tilde{p}\| \leq 2^{q-2}\, [V^{q-2}(p)\, \|p\| + V^{q-2}(\tilde{p})\, \|\tilde{p}\|].$$

Now let $1 \leq q < 2$.

Setting $F(t) = (1 + t^2)^{\frac{q-2}{2}} \cdot t$, we have

$$F'(t) = (1 + t^2)^{\frac{q-4}{2}}\, [1 + t^2(q - 1)] \geq 0$$

then $F(t)$ is a nondecreasing function.

Hence, if $\|\tilde{p}\| \leq \|p\|$, we have

$$F(\|p + \tilde{p}\|) \leq F(2\|p\|)$$

that is

$$\begin{aligned}V^{q-2}(p + \tilde{p})\, \|p + \tilde{p}\| = \left(1 + \|p + \tilde{p}\|^2\right)^{\frac{q-2}{2}}\, \|p + \tilde{p}\| &\leq 2\left(1 + 4\|p\|^2\right)^{\frac{q-2}{2}}\, \|p\| \\ &\leq 2\, \left(1 + \|p\|^2\right)^{\frac{q-2}{2}}\, \|p\|.\end{aligned}$$

If $\|p\| \leq \|\tilde{p}\|$, in a similar way, we obtain

$$V^{q-2}(p + \tilde{p})\, \|p + \tilde{p}\| \leq 2\, V^{q-2}(\tilde{p})\, \|\tilde{p}\|.$$

Then, by summing the previous inequalities, we obtain

$$V^{q-2}(p + \tilde{p})\, \|p + \tilde{p}\| \leq V^{q-2}(p)\, \|p\| + V^{q-2}(\tilde{p})\, \|\tilde{p}\|.$$

Finally we recall that, if $G \in L^q(\Omega)$, $q > 1$, it easily follows:

$$\|G\|^q_{L^q(\Omega)} \leq \|W(G)\|^2_{L^2(\Omega)}. \tag{9}$$

## 3 Proof of Theorem 1

In order to obtain the global higher summability of the gradient, we prove in a first step the interior higher summability of the gradient. To this end a crucial step is the following "Caccioppoli's type" inequality.

Let us start with the case $q \geq 2$ and let us recall the proof given in [15] in order to specify the constants which appear in estimate (11).

**Theorem 3** *Assume that conditions (4)–(7) are fulfilled. Let $g \in H^{1,s}(\Omega) \cap L^{\infty}(\Omega)$, $s > q$, and let $w \in H_0^{1,q}(\Omega) \cap L^{\infty}(\Omega)$, $q \geq 2$, be a solution of the strongly elliptic system:*

$$\int_{\Omega} \sum_{i=1}^{n} \left( a^i(x, w+g, Dw+Dg) | D_i \varphi \right) \mathrm{d}x = \int_{\Omega} (B(x, w+g, Dw+Dg) | \varphi) \, \mathrm{d}x,$$
$$\forall \varphi \in H_0^{1,q}(\Omega) \cap L^{\infty}(\Omega). \tag{10}$$

*Then for every couples of concentric balls $B(x^0, \sigma) \subset B(x^0, 2\sigma) \subset \Omega$, it results*

$$\int_{B(x^0,\sigma)} \|Dw\|^q \mathrm{d}x \leq c\, \sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x + c_1 \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x, \tag{11}$$

*where* $c = \dfrac{q\, 2^{q+1} n M \gamma^q}{(\nu - 2b\|u-g\|_{L^{\infty}(\Omega)})} \left( \dfrac{nM(1+8q)\, 2^{3q-2}}{\nu - 2b\|u-g\|_{L^{\infty}(\Omega)}} \right)^{q-1}$, $c_1 = 2^{q+1} + 8\dfrac{q+2^{-q}}{1+8q} + \dfrac{2^{q+2}\, a\|u-g\|_{L^{\infty}(\Omega)}}{\nu - 2b\|u-g\|_{L^{\infty}(\Omega)}} + nM \dfrac{2^{3q-3}}{\nu - 2b\|u-g\|_{L^{\infty}(\Omega)}} \left[ 4 + \left( \dfrac{nM(1+8q)2^{3q-2}}{\nu - 2b\|u-g\|_{L^{\infty}(\Omega)}} \right)^{q-1} + \left( \dfrac{nM(1+8q)2^{3q-2}}{\nu - 2b\|u-g\|_{L^{\infty}(\Omega)}} \right)^{\frac{1}{q-1}} \right]$ *and $\gamma$ is a numerical constant.*

*Proof* Let us fix $B(x^0, 2\sigma) \subset \Omega$ and let $\theta \in C_0^{\infty}(\mathbb{R}^n)$ be a function with these properties

$$0 \leq \theta \leq 1, \quad \theta = 1 \text{ in } B(x^0, \sigma), \quad \theta = 0 \text{ in } \mathbb{R}^n \setminus B(x^0, 2\sigma), \quad \|D\theta\| \leq \gamma \sigma^{-1}$$

with $\gamma$ numerical constant. Let us assume in (10) $\varphi = \theta^q (w - w_{B(x^0,2\sigma)})$. Then (10) becomes:

$$\int_{\Omega} \sum_{i=1}^{n} \left( a^i(x, w+g, Dw+Dg) \,|\, \theta^q D_i w \right) \mathrm{d}x$$
$$= -q \int_{\Omega} \sum_{i=1}^{n} a^i(x, w+g, Dw+Dg) \,|\, \theta^{q-1} D_i\theta \; (w - w_{B(x^0,2\sigma)}) \, \mathrm{d}x$$
$$+ \int_{\Omega} \left( B(x, w+g, Dw+Dg) \,|\, \theta^q \, (w - w_{B(x^0,2\sigma)}) \right) \mathrm{d}x. \tag{12}$$

We may rewrite (12) in the equivalent way

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^n \Big(a^i(x, w+g, Dw+Dg) \,|\, \theta^q\,(D_i w + D_i g)\Big)\,\mathrm{d}x \\
&\quad = \int_\Omega \sum_{i=1}^n \Big(a^i(x, w+g, Dw+Dg) \,|\, \theta^q\, D_i g\Big)\,\mathrm{d}x \\
&\qquad -q\int_\Omega \sum_{i=1}^n a^i(x, w+g, Dw+Dg) \,|\, \theta^{q-1} D_i\theta\,(w - w_{B(x^0,2\sigma)})\,\mathrm{d}x \\
&\qquad + \int_\Omega \big(B(x, w+g, Dw+Dg) \,|\, \theta^q\,(w - w_{B(x^0,2\sigma)})\big)\,\mathrm{d}x = A + C + D. \qquad (13)
\end{aligned}$$

As it concerns the left hand side of (13), in virtue of the strong ellipticity condition (5), it results

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^n \Big(a^i(x, w+g, Dw+Dg) \,|\, \theta^q\,(D_i w + D_i g)\Big)\,\mathrm{d}x \\
&\quad \geq \nu \int_\Omega \|Dw + Dg\|^2\; V^{q-2}(Dw+Dg)\,\theta^q \mathrm{d}x \\
&\quad = \nu \int_\Omega \|W(Dw+Dg)\|^2\,\theta^q \mathrm{d}x. \qquad (14)
\end{aligned}$$

Let us examine the terms in the right hand side of (13) and let us start with the first term $A$. By condition (4), taking into account Lemma 2, we get

$$\begin{aligned}
|A| &\leq \int_\Omega \sum_{i=1}^n \|a^i(x, w+g, Dw+Dg)\|\,\theta^q\,\|D_i g\|\mathrm{d}x \\
&\leq nM \int_\Omega V^{q-2}(Dw+Dg)\|Dw+Dg\|\,\theta^q\,\|Dg\|\,\mathrm{d}x \\
&\leq nc(q)M \int_\Omega V^{q-2}(Dw)\|Dw\|\,\theta^q\,\|Dg\|\,\mathrm{d}x + nc(q)M\int_\Omega V^{q-2}(Dg)\|Dg\|^2\,\theta^q\,\mathrm{d}x \\
&\leq n2^{q-2}c(q)M \int_\Omega \theta^q\,(1 + \|Dw\|^{q-2})\,\|Dw\|\,\|Dg\|\,\mathrm{d}x \\
&\quad + nc(q)M \int_\Omega \theta^q (1 + \|Dg\|^2)^{\frac{q-2}{2}}\,(1 + \|Dg\|^2)\mathrm{d}x \\
&\leq n2^{q-2}c(q)M \int_\Omega \theta^q(\|Dw\|\,\|Dg\| + \|Dw\|^{q-1}\|Dg\|)\mathrm{d}x \\
&\quad + nc(q)M \int_\Omega \theta^q (1 + \|Dg\|^2)^{\frac{q}{2}}\,\mathrm{d}x, \qquad (15)
\end{aligned}$$

where $c(q) = 2^{q-2}$.

Using Young's inequality in the integral in the penultimate line of (15), it follows

$$|A| \le 2^{2q-4} nM \left[ 2\varepsilon \int_\Omega \theta^q \|Dw\|^q \mathrm{d}x + \left( \varepsilon^{-\frac{1}{q-1}} + \varepsilon^{-(q-1)} \right) \int_\Omega \theta^q (1 + \|Dg\|)^q \mathrm{d}x \right]$$
$$+ 2^{2q-2} nM \int_\Omega \theta^q (1 + \|Dg\|^q) \mathrm{d}x \le 2^{2q-3} nM\varepsilon \int_\Omega \theta^q \|Dw\|^q \mathrm{d}x$$
$$+ 2^{2q-4} nM \left( \varepsilon^{-\frac{1}{q-1}} + \varepsilon^{-(q-1)} + 4 \right) \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x. \tag{16}$$

Let us consider the second term C. In virtue of condition (4)

$$|C| \le q \int_\Omega \sum_{i=1}^n \|a^i(x, w+g, Dw+Dg)\| \, \theta^{q-1} \, |D_i\theta| \|w - w_{B(x^0,2\sigma)}\| \, \mathrm{d}x$$
$$\le nqM \int_\Omega V^{q-2}(Dw+Dg) \|Dw+Dg\| \, \theta^{q-1} \, \|D\theta\| \|w - w_{B(x^0,2\sigma)}\| \, \mathrm{d}x$$
$$\le nqM \int_\Omega (1 + \|Dw+Dg\|)^{q-2} \, \|Dw+Dg\| \, \theta^{q-1} \, \|D\theta\| \|w - w_{B(x^0,2\sigma)}\| \, \mathrm{d}x$$
$$\le nqM \int_\Omega (1 + \|Dw\| + \|Dg\|)^{q-1} \, \theta^{q-1} \, \|D\theta\| \|w - w_{B(x^0,2\sigma)}\| \, \mathrm{d}x. \tag{17}$$

Applying Hölder inequality, we get

$$|C| \le nqM\varepsilon \int_\Omega (1 + \|Dw\| + \|Dg\|)^q \, \theta^q \, \mathrm{d}x + \frac{nqM}{\varepsilon^{q-1}} \int_\Omega \|D\theta\|^q \|w - w_{B(x^0,2\sigma)}\|^q \, \mathrm{d}x$$
$$\le 2^q nqM\varepsilon \int_\Omega (1 + \|Dw\|)^q \, \theta^q \, \mathrm{d}x + 2^q nqM\varepsilon \int_\Omega \theta^q \|Dg\|^q \, \mathrm{d}x$$
$$+ \frac{nqM\gamma^q}{\varepsilon^{q-1}} \sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \, \mathrm{d}x \le 2^q nqM\varepsilon 2^q \int_{B(x^0,2\sigma)} \theta^q \, \mathrm{d}x$$
$$+ 2^q nqM\varepsilon 2^q \int_\Omega \|Dw\|^q \, \theta^q \, \mathrm{d}x + 2^q nqM\varepsilon \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \, \mathrm{d}x$$
$$+ \frac{nqM\gamma^q}{\varepsilon^{q-1}} \sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \, \mathrm{d}x. \tag{18}$$

Finally for the last term $D$, we have from condition (6)

$$|D| \le \int_\Omega \|B(x, w+g, Dw+Dg)\| \, \theta^q \, \|w - w_{B(x^0,2\sigma)}\| \mathrm{d}x$$
$$\le 2 \int_\Omega (a + b\|W(Dw+Dg)\|^2) \, \theta^q \, \|w\|_{L^\infty(\Omega)} \, \mathrm{d}x$$
$$= 2a \int_\Omega \theta^q \|w\|_{L^\infty(\Omega)} \mathrm{d}x + 2b \|w\|_{L^\infty(\Omega)} \int_\Omega \|W(Dw+Dg)\|^2 \, \theta^q \mathrm{d}x. \tag{19}$$

Taking into account (13), (14), (16), (18), (19), we get

$$\begin{aligned}
&(\nu - 2b\|w\|_{L^\infty(\Omega)}) \int_\Omega \|W(Dw + Dg)\|^2\, \theta^q \mathrm{d}x \\
&\quad \leq (2^{2q-3} nM\varepsilon + 2^{2q} qnM\varepsilon) \int_\Omega \|Dw\|^q\, \theta^q\, \mathrm{d}x \\
&\qquad + (2^{2q} nqM\varepsilon + 2a\|u - g\|_{L^\infty(\Omega)}) \int_{B(x^0,2\sigma)} \theta^q\, \mathrm{d}x \\
&\qquad + \left[2^{2q-4} nM \left(4 + \varepsilon^{-(q-1)} + \varepsilon^{-\frac{1}{q-1}}\right) + nqM2^q\varepsilon\right] \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x \\
&\qquad + \frac{nqM\gamma^q}{\varepsilon^{q-1}} \sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q\, \mathrm{d}x.
\end{aligned} \tag{20}$$

Since, in virtue of (9),

$$\int_\Omega \theta^q \|Dw\|^q\, \mathrm{d}x \leq 2^q \int_\Omega \|W(Dw + Dg)\|^2\, \theta^q \mathrm{d}x + 2^q \int_\Omega (1 + \|Dg\|)^q \theta^q \mathrm{d}x$$

and

$$\int_{B(x^0,2\sigma)} \theta^q\, \mathrm{d}x \leq \int_{B(x^0,2\sigma)} \mathrm{d}x \leq \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x \tag{21}$$

we get

$$\begin{aligned}
&(\nu - 2b\|w\|_{L^\infty(\Omega)}) \int_\Omega \|W(Dw + Dg)\|^2\, \theta^q \mathrm{d}x \\
&\quad \leq 2^{3q-3}(1 + 8q)nM\varepsilon \int_\Omega \|W(Dw + Dg)\|^2\, \theta^q \mathrm{d}x \\
&\qquad + \Big[2^{3q-3}(1 + 8q)nM\varepsilon + 2^{2q} qnM\varepsilon + 2a\|u - g\|_{L^\infty(\Omega)} \\
&\qquad + 2^{2q-4} nM \left(4 + \varepsilon^{-(q-1)} + \varepsilon^{-\frac{1}{q-1}}\right) + 2^q qnM\varepsilon\Big] \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x \\
&\qquad + \frac{nqM\gamma^q}{\varepsilon^{q-1}} \sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q\, \mathrm{d}x.
\end{aligned}$$

Choosing $\varepsilon^*$ in such a way that

$$\frac{1}{2}(\nu - 2b\|w\|_{L^\infty(\Omega)}) = 2^{3q-3}(1 + 8q)nM\varepsilon^*$$

from (7), (9) and (20) we obtain

$$\int_{B(x^0,\sigma)} \|Dw\|^q \,\mathrm{d}x$$
$$\le \frac{q\ 2^{q+1} nM\gamma^q}{(\nu - 2b\|u-g\|_{L^\infty(\Omega)})} \left(\frac{nM(1+8q)\,2^{3q-2}}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}\right)^{q-1}$$
$$\cdot\sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \,\mathrm{d}x + \Bigg\{ 2^{q+1} + 8\frac{q+2^{-q}}{1+8q}$$
$$+ \frac{2^{q+2}\ a\|u-g\|_{L^\infty(\Omega)}}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}} + nM \frac{2^{3q-3}}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}} \Bigg[ 4 + \left(\frac{nM(1+8q)2^{3q-2}}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}\right)^{q-1}$$
$$+ \left(\frac{nM(1+8q)2^{3q-2}}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}\right)^{\frac{1}{q-1}} \Bigg] \Bigg\} \int_{B(x^0,2\sigma)} (1+\|Dg\|)^q \mathrm{d}x \tag{22}$$

that is our thesis.

We are in position to derive the interior higher summability of the gradient.

**Theorem 4** *Assume that conditions (4)–(7) are fulfilled. Let $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, $s > q$, and let $w \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $q \ge 2$, be a solution of the strongly elliptic system*

$$\int_\Omega \sum_{i=1}^n \left(a^i(x, w+g, Dw+Dg)|D_i\varphi\right) \mathrm{d}x$$
$$= \int_\Omega (B(x, w+g, Dw+Dg)|\varphi)\,\mathrm{d}x, \qquad \forall\varphi \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega), \tag{23}$$

*then there exists a number $\tilde r > 1$ such that $Du \in L^{q\tilde r}_{\mathrm{loc}}(\Omega)$ and $\forall B(x^0, 2\sigma) \subset \Omega$ it results*

$$\left(\fint_{B(x^0,\sigma)} \|Dw\|^{q\tilde r} \mathrm{d}x\right)^{\frac{1}{\tilde r}}$$
$$\le K \fint_{B(x^0,2\sigma)} \|Dw\|^q \mathrm{d}x + K\left(\fint_{B(x^0,2\sigma)} (1+\|Dg\|)^{q\tilde r}\,\mathrm{d}x\right)^{\frac{1}{\tilde r}} \tag{24}$$

*where the constant $K$ does not depend on $\sigma$.*

*Proof* Let us recall the Poincaré inequality related to a function $u \in H^{1,p}(\Omega)$, $p < n$:

$$\left(\int_\Omega \|u - u_\Omega\|^{\frac{np}{n-p}} \mathrm{d}x\right)^{\frac{n-p}{np}} \le c(n,N)\frac{(n-1)p}{n-p}\left(\int_\Omega \|Du\|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

where the dependence on $p$ of the constant has been pointed out. By choosing $p = \frac{nq}{n-q} < n$, we get for the function $w \in H^{1,\frac{nq}{n+q}}(\Omega)$ solution of the strongly elliptic system (23):

$$\sigma^{-q}\int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x \le \tilde c(n,N)(2q)^q\sigma^n \left(\fint_{B(x^0,2\sigma)} \|Dw\|^{\frac{nq}{n+q}} \mathrm{d}x\right)^{\frac{n+q}{n}}.$$

Hence if we set

$$U = \|Dw\|^{\frac{qn}{n+q}}, \qquad G = (1+\|Dg\|)^{\frac{qn}{n+q}}$$

from "Caccioppoli's inequality" (11) it follows

$$\fint_{B(x^0,\sigma)} U^{\frac{n+q}{n}} \mathrm{d}x \le c\,\tilde{c}(n) q 2^q \left(\fint_{B(x^0,2\sigma)} U \mathrm{d}x\right)^{\frac{n+q}{n}} + c_1\, 2^n \fint_{B(x^0,2\sigma)} G^{\frac{n+q}{n}} \mathrm{d}x$$

with $c$ and $c_1$ as in Theorem 3.1.

Then, in virtue of Lemma 1, the assert follows.

In the second step of the proof of the global higher summability, we have to prove the higher summability up to the boundary for the gradient of a solution to Dirichlet Problem.

**Theorem 5** *Let conditions* (4)–(7) *be fulfilled and* $g \in H^{1,s}(B^+(1)) \cap L^\infty(B^+(1))$, *with* $s > q \ge 2$. *If* $w \in H^{1,q}(B^+(1)) \cap L^\infty(B^+(1))$ *is a solution of the strongly elliptic problem*

$$\begin{cases} \int_{B^+(1)} \sum_{i=1}^{n} \left(a^i(x, w+g, Dw+Dg) | D_i\varphi\right) \mathrm{d}x \\ = \int_{B^+(1)} (B(x, w+g, Dw+Dg)|\varphi)\, \mathrm{d}x, \quad \forall \varphi \in H_0^{1,q}(B^+(1)) \cap L^\infty(B^+(1)) \\ w(x) = 0 \quad on\ \Gamma, \end{cases} \tag{25}$$

*then there exists a number* $r' > 1$ *such that* $Dw \in L^{qr'}_{\mathrm{loc}}(B^+(1))$ *and for all* $B(x^0, 2\sigma) \subset B(1)$ *such that* $B(x^0, 2\sigma) \cap B^+(1) \ne \varnothing$ *it results*

$$\left(\fint_{B(x^0,\sigma)\cap B^+(1)} \|Dw\|^{qr'} \mathrm{d}x\right)^{\frac{1}{r'}} \le K \fint_{B(x^0,2\sigma)\cap B^+(1)} \|Dw\|^q \mathrm{d}x + K\left(\fint_{B(x^0,2\sigma)\cap B^+(1)} (1+\|Dg\|)^{qr'}\, \mathrm{d}x,\right)^{\frac{1}{r'}} \tag{26}$$

*where* $K$ *is a positive constant that does not depend on* $\sigma$.

*Proof* First let us show that for every $B(x^0, 2\sigma) \subset B(1)$ such that $B(x^0, 2\sigma) \cap B^+(1) \ne \varnothing$ it results

$$\int_{B(x^0,\sigma)\cap B^+(1)} \|Dw\|^q \mathrm{d}x \le c\sigma^n \left(\fint_{B(x^0,2\sigma)\cap B^+(1)} \|Dw\|^{\frac{nq}{n+q}} \mathrm{d}x\right)^{\frac{n+q}{n}} + c_1 \int_{B(x^0,2\sigma)\cap B^+(1)} (1+\|Dg\|)^q\, \mathrm{d}x. \tag{27}$$

We must consider two cases.

First let $B(x^0, 2\sigma) \subset B^+(1)$, namely it results $x_n^0 > 2\sigma$. Arguing as in the proof of Theorem 3 we get the estimate:

$$\int_{B(x^0,\sigma)} \|Dw\|^q \mathrm{d}x \le c\,\sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x + c_1 \int_{B(x^0,2\sigma)} (1+\|Dg\|)^q \mathrm{d}x \tag{28}$$

with $c$ and $c_1$ constants as in Theorem 3.

In the second case let $x^0 \in B^+(1)$ such that $x_n^0 < 2\sigma$ and hence $B(x^0, 2\sigma) \cap B^+(1) \ne \varnothing$. Let us choose a function $\theta \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \le \theta \le 1, \quad \theta = 1 \text{ in } B(x^0,\sigma), \quad \theta = 0 \text{ in } \mathbb{R}^n \setminus B(x^0, 2\sigma), \quad \|D\theta\| \le C\sigma^{-1}$$

with $C$ numerical constant. Taking into account that $w = 0$ on $\Gamma$, in (25) we can assume $\varphi = \theta^q\, w$ and, arguing again as in the proof of Theorem 3, we get

$$\int_{B(x^0,\sigma)\cap B^+(1)} \|Dw\|^q \mathrm{d}x \le c\,\sigma^{-q} \int_{B(x^0,2\sigma)\cap B^+(1)} \|w\|^q \mathrm{d}x + c_1 \int_{B(x^0,2\sigma)\cap B^+(1)} (1+\|Dg\|)^q\, \mathrm{d}x \tag{29}$$

with $c$ and $c_1$ constants as in Theorem 3. Of course if $B(x^0,\sigma)\cap B^+(1) = \varnothing$, the left-hand side is zero.

From (28) we get (see the proof of Theorem 4)

$$\int_{B(x^0,\sigma)} \|Dw\|^q \mathrm{d}x \le c\tilde{c}(n,N)(2q)^q\, \sigma^n \left( \fint_{B(x^0,2\sigma)} \|Dw\|^{\frac{nq}{n+q}} \mathrm{d}x \right)^{\frac{n+q}{n}} + c_1 \int_{B(x^0,2\sigma)} (1+\|Dg\|)^q\, \mathrm{d}x.$$

From (29), taking into account that $w(x) \equiv 0$ on $\Gamma$ and applying the Poincaré inequality

$$\sigma^{-q} \int_{B(x^0,2\sigma)\cap B^+(1)} \|w\|^q \mathrm{d}x \le \tilde{c}(n,N)(2q)^q\, \sigma^n \left( \fint_{B(x^0,2\sigma)\cap B^+(1)} \|Dw\|^{\frac{nq}{n+q}} \mathrm{d}x \right)^{\frac{n+q}{n}}$$

we obtain

$$\int_{B(x^0,\sigma)\cap B^+(1)} \|Dw\|^q \mathrm{d}x \le c\tilde{c}(n,N)(2q)^q\, \sigma^n \left( \fint_{B(x^0,2\sigma)\cap B^+(1)} \|Dw\|^{\frac{nq}{n+q}} \mathrm{d}x \right)^{\frac{n+q}{n}} + c_1 \int_{B(x^0,2\sigma)\cap B^+(1)} (1+\|Dg\|)^q\, \mathrm{d}x$$

and hence the estimate (27) is proved.

Now let us consider the function $\tilde{w} \in H_0^1(B(x^0,\sigma))$ and $h \in L^s(B(1))$ obtained from $w$ and $\|Dg\|$ by setting

$$\tilde{w}(x) = \begin{cases} w(x), & \text{if } x_n \ge 0 \\ 0, & \text{if } x_n < 0 \end{cases}, \quad h(x) = \begin{cases} \|Dg(x)\|, & \text{if } x_n \ge 0, \\ 0, & \text{if } x_n < 0 \end{cases}$$

and let us remark that for $B(x^0, 2\sigma) \subset B(1)$ we have

$$\begin{aligned} \int_{B(x^0,\sigma)} \|D\tilde{w}\|^q \mathrm{d}x &= \int_{B(x^0,\sigma)\cap B^+(1)} \|Dw\|^q \mathrm{d}x \\ &\le c\tilde{c}(n,N)(2q)^q\, \sigma^n \left( \fint_{B(x^0,2\sigma)\cap B^+(1)} \|Dw\|^{\frac{nq}{n+q}} \mathrm{d}x \right)^{\frac{n+q}{n}} \\ &\quad + c_1 \int_{B(x^0,2\sigma)\cap B^+(1)} (1+\|Dg\|)^q\, \mathrm{d}x \\ &\le c\tilde{c}(n,N)(2q)^q\, \sigma^n \left( \fint_{B(x^0,2\sigma)} \|D\tilde{w}\|^{\frac{nq}{n+q}} \mathrm{d}x \right)^{\frac{n+q}{n}} \\ &\quad + c_1 \int_{B(x^0,2\sigma)} (1+h)^q\, \mathrm{d}x, \end{aligned}$$

where of course if $B(x^0,\sigma)\cap B^+(1) = \varnothing$, the integrals are zero.

Then the functions $\tilde{w}$ and $h$ fulfill all the assumptions of Lemma 1 and, arguing as in the proof of Theorem 4, we get that there exists a number $r' > 1$ such that $D\tilde{w} \in L^{qr'}_{\text{loc}}(B(1))$ and $\forall B(x^0, 2\sigma) \subset B(1)$ it results:

$$\left(\fint_{B(x^0,\sigma)} \|D\tilde{w}\|^{qr'} \mathrm{d}x\right)^{\frac{1}{r'}}$$
$$\leq K \fint_{B(x^0,2\sigma)} \|D\tilde{w}\|^q \mathrm{d}x + K \left(\fint_{B(x^0,2\sigma)} (1+h)^{qr'}\,\mathrm{d}x\right)^{\frac{1}{r'}}$$

and hence the assert.

Now we may derive the global higher summability of the gradient.

**Theorem 6** *Let conditions (4)–(7) be fulfilled. Let $\partial\Omega$ be of class $C^2$ and $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, with $s > q$. If $w \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $q \geq 2$, is a solution to the Dirichlet problem*

$$\begin{cases} \sum_{i=1}^n D_i a^i(x, w+g, Dw+Dg) = -B(x, w+g, Dw+Dg) \\ w = 0 \text{ on } \partial\Omega, \end{cases}$$

*then there exists $r > 1$ such that $Dw \in L^{qr}(\Omega)$.*

*Proof* Taking into account that $\partial\Omega$ is of class $C^2$, it is enough to use the usual covering procedure (see [5] Lemma 2.IV, 2.V and Section n.8 for details).

## 4 Proof of theorem 2

Also in the case $1 < q < 2$ a crucial step, in order to obtain the global higher summability of the gradient, is to show a Caccioppoli's type inequality. However in this case the procedure is different in virtue of the effect of degeneration.

**Theorem 7** *Assume that conditions (4)–(7) are fulfilled. Let $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, $s > q$, and let $w \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $1 < q < 2$, be a solution of the strongly elliptic system:*

$$\int_\Omega \sum_{i=1}^n \left(a^i(x, w+g, Dw+Dg)|D_i\varphi\right) \mathrm{d}x = \int_\Omega (B(x, w+g, Dw+Dg)|\varphi)\,\mathrm{d}x,$$
$$\forall \varphi \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega). \tag{30}$$

*Then for every couples of concentric balls $B(x^0,\sigma) \subset B(x^0, 2\sigma) \subset \Omega$, it results*

$$\int_{B(x^0,\sigma)} \|Dw\|^q \mathrm{d}x \leq c\,\sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x$$
$$+c_1 \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x \tag{31}$$

*where* $c = \dfrac{270336 n^2 M^2 (1+\gamma^2)}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}$, $c_1 = \dfrac{16a\|u-g\|_{L^\infty(\Omega)}}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}} + \dfrac{16Mn}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}} + \dfrac{8Mn}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}\left(1 + \dfrac{16896nM}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}\right) + \dfrac{337}{2112}$.

*Proof* Arguing in a similar way as in Theorem 3, we obtain

$$\int_\Omega \sum_{i=1}^n \Big(a^i(x, w+g, Dw+Dg)\,|\,\theta^q\,(D_i w + D_i g\Big)\,\mathrm{d}x$$
$$= \int_\Omega \sum_{i=1}^n \Big(a^i(x, w+g, Dw+Dg)\,|\,\theta^q\, D_i g\Big)\,\mathrm{d}x$$
$$-q\int_\Omega \sum_{i=1}^n a^i(x, w+g, Dw+Dg)\,|\,\theta^{q-1} D_i\theta\,(w - w_{B(x^0,2\sigma)})\,\mathrm{d}x$$
$$+\int_\Omega \big(B(x, w+g, Dw+Dg)\,|\,\theta^q\,(w - w_{B(x^0,2\sigma)}\big)\,\mathrm{d}x = A + C + D. \quad (32)$$

and

$$\int_\Omega \sum_{i=1}^n \Big(a^i(x, w+g, Dw+Dg)\,|\,\theta^q\,(D_i w + D_i g\Big)\,\mathrm{d}x$$
$$\geq \nu \int_\Omega \|Dw+Dg\|^2\, V^{q-2}(Dw+Dg)\,\theta^q \mathrm{d}x$$
$$= \nu \int_\Omega \|W(Dw+Dg)\|^2\,\theta^q \mathrm{d}x. \quad (33)$$

Let us consider the first term at the right hand side of (32). By condition (4) and Lemma 2

$$|A| \leq \int_\Omega \sum_{i=1}^n \|a^i(x, w+g, Dw+Dg)\|\,\theta^q\,\|D_i g\|\mathrm{d}x$$
$$\leq nM \int_\Omega V^{q-2}(Dw+Dg)\|Dw+Dg\|\,\theta^q\,\|Dg\|\,\mathrm{d}x$$
$$\leq nM \int_\Omega V^{q-2}(Dw)\|Dw\|\,\theta^q\,\|Dg\|\,\mathrm{d}x + nM\int_\Omega V^{q-2}(Dg)\|Dg\|^2\,\theta^q\,\mathrm{d}x$$
$$\leq nM \int_\Omega \theta^q\,(1+\|Dw\|^{q-1})\,\|Dg\|\,\mathrm{d}x + nM\int_\Omega \theta^q(1+\|Dg\|^2)^{\frac{q-2}{2}}\,(1+\|Dg\|^2)\mathrm{d}x$$
$$\leq nM \int_\Omega \theta^q(1+\|Dg\|)^q\mathrm{d}x + nM\int_\Omega \theta^q\|Dw\|^{q-1}\|Dg\|\mathrm{d}x$$
$$+nM\int_\Omega \theta^q(1+\|Dg\|^2)^{\frac{q}{2}}\mathrm{d}x$$
$$\leq 2nM \int_\Omega \theta^q(1+\|Dg\|)^q\mathrm{d}x + nM\int_\Omega \theta^q\|Dw\|^{q-1}\|Dg\|\mathrm{d}x. \quad (34)$$

Using Young's inequality in the second integral in the last line of (34), it follows

$$|A| \leq 2nM\int_{B(x^0,2\sigma)} (1+\|Dg\|)^q\mathrm{d}x + n\varepsilon M\int_\Omega \theta^q\|Dw\|^q\mathrm{d}x$$
$$+\frac{nM}{\varepsilon^{q-1}}\int_{B(x^0,2\sigma)} \theta^q\|Dg\|^q\mathrm{d}x$$
$$\leq \left(2+\frac{1}{\varepsilon^{q-1}}\right) nM\int_{B(x^0,2\sigma)} (1+\|Dg\|)^q\mathrm{d}x + n\varepsilon M\int_\Omega \theta^q\|Dw\|^q\mathrm{d}x. \quad (35)$$

Let us examine the second term $C$ in (32)

$$
\begin{aligned}
|C| &\le q \int_\Omega \sum_{i=1}^{n} \|a^i(x, w+g, Dw+Dg)\|\, \theta^{q-1}\, |D_i\theta| \|w - w_{B(x^0,2\sigma)}\| \,\mathrm{d}x \\
&\le nqM \int_\Omega V^{q-2}(Dw+Dg)\|Dw+Dg\|\, \theta^{q-1}\, \|D\theta\| \|w - w_{B(x^0,2\sigma)}\| \,\mathrm{d}x \\
&= nqM \int_\Omega (1+\|Dw+Dg\|^2)^{\frac{q-2}{2}}\, \|Dw+Dg\|\, \theta^{q-1}\, \|D\theta\| \|w - w_{B(x^0,2\sigma)}\| \,\mathrm{d}x \\
&\le nqM \int_\Omega (1+\|Dw+Dg\|^2)^{\frac{q-1}{2}}\, \theta^{q-1}\, \|D\theta\| \|w - w_{B(x^0,2\sigma)}\| \,\mathrm{d}x \\
&\le 2Mn \int_\Omega (1+\|Dw\|+\|Dg\|)^{q-1}\, \theta^{q-1}\, \|D\theta\| \|w - w_{B(x^0,2\sigma)}\| \,\mathrm{d}x. 
\end{aligned} \tag{36}
$$

Applying Young's inequality, we reach

$$
\begin{aligned}
|C| &\le 2Mn\varepsilon \int_\Omega (1+\|Dw\|+\|Dg\|)^q\, \theta^q \,\mathrm{d}x + \frac{2nM}{\varepsilon^{q-1}} \int_\Omega \|D\theta\|^q \|w - w_{B(x^0,2\sigma)}\|^q \,\mathrm{d}x \\
&\le 8Mn\varepsilon \int_\Omega (1+\|Dw\|)^q\, \theta^q \,\mathrm{d}x \\
&\quad + 8Mn\varepsilon \int_{B(x^0,2\sigma)} \|Dg\|^q \,\mathrm{d}x + \frac{2nM\gamma^q}{\varepsilon^{q-1}}\sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \,\mathrm{d}x \\
&\le 32Mn\varepsilon \int_{B(x^0,2\sigma)} \theta^q \,\mathrm{d}x + 32Mn\varepsilon \int_\Omega \|Dw\|^q\, \theta^q \,\mathrm{d}x \\
&\quad + 8Mn\varepsilon \int_{B(x^0,2\sigma)} (1+\|Dg\|)^q \,\mathrm{d}x \\
&\quad + \frac{2nM\gamma^q}{\varepsilon^{q-1}}\sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \,\mathrm{d}x.
\end{aligned} \tag{37}
$$

Finally for the last term $D$, we have from condition (6)

$$
\begin{aligned}
|D| &\le \int_\Omega \|B(x, w+g, Dw+Dg)\|\, \theta^q\, \|w - w_{B(x^0,2\sigma)}\| \mathrm{d}x \\
&\le 2 \int_\Omega (a + b\|W(Dw+Dg)\|^2)\, \theta^q \|w\|_{L^\infty(\Omega)} \,\mathrm{d}x \\
&= 2a \int_\Omega \theta^q \|w\|_{L^\infty(\Omega)} \mathrm{d}x + 2b\|w\|_{L^\infty(\Omega)} \int_\Omega \|W(Dw+Dg)\|^2\, \theta^q \mathrm{d}x.
\end{aligned} \tag{38}
$$

Taking into account (21), (32), (33), (35), (37), (38), we obtain

$$
\begin{aligned}
&(\nu - 2b\|w\|_{L^\infty(\Omega)}) \int_\Omega \|W(Dw+Dg)\|^2\, \theta^q \mathrm{d}x \\
&\quad \le 33Mn\varepsilon \int_\Omega \|Dw\|^q\, \theta^q \,\mathrm{d}x + \left(2a\|u-g\|_{L^\infty(\Omega)} + 40Mn\varepsilon + 2Mn + \frac{nM}{\varepsilon^{q-1}}\right) \\
&\quad\quad \times \int_{B(x^0,2\sigma)} (1+\|Dg\|)^q \mathrm{d}x + \frac{2nM\gamma^q}{\varepsilon^{q-1}}\sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \,\mathrm{d}x.
\end{aligned} \tag{39}
$$

As it concerns the term $\int_\Omega \|Dw\|^q\, \theta^q \,\mathrm{d}x$, it results

$$
\|Dw\|^q\, \theta^q \le 4\theta^q (\|Dw+Dg\|^q + \|Dg\|^q)
$$

and

$$\theta^q \|Dw + Dg\|^q$$
$$= \left(\theta^{\frac{q^2}{2}} (V(Dw + Dg))^{(q-2)\frac{q}{2}} \|Dw + Dg\|^q\right) \left(\theta^{q-\frac{q^2}{2}} (V(Dw + Dg))^{(2-q)\frac{q}{2}}\right).$$

Applying Young's inequality, we obtain for each $0 < \eta < 1$

$$\int_\Omega \|Dw\|^q \theta^q \, \mathrm{d}x$$
$$\leq 4 \int_\Omega \theta^q \|Dw + Dg\|^q \mathrm{d}x + 4 \int_\Omega \theta^q \|Dg\|^q \mathrm{d}x$$
$$\leq 4 \int_\Omega \theta^q \|Dg\|^q \mathrm{d}x + \frac{4}{\eta} \int_\Omega \left[\theta^{\frac{q^2}{2}} (V(Dw + Dg))^{(q-2)\frac{q}{2}} \|Dw + Dg\|^q\right]^{\frac{2}{q}} \mathrm{d}x$$
$$+4\eta \int_\Omega \left[\theta^{q-\frac{q^2}{2}} (V(Dw + Dg))^{(2-q)\frac{q}{2}}\right]^{\frac{2}{2-q}} \mathrm{d}x \leq 4 \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x$$
$$+\frac{4}{\eta} \int_\Omega \theta^q \|W(Dw + Dg)\|^2 \mathrm{d}x + 4\eta \int_\Omega \theta^q \left(1 + \|Dw + Dg\|^2\right)^{\frac{q}{2}} \mathrm{d}x$$
$$\leq 4 \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x + \frac{4}{\eta} \int_\Omega \theta^q \|W(Dw + Dg)\|^2 \mathrm{d}x$$
$$+16\eta \int_\Omega \theta^q \|Dw\|^q \mathrm{d}x + 16\eta \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x.$$

For $\eta = \frac{1}{32}$ we get

$$\int_\Omega \|Dw\|^q \theta^q \, \mathrm{d}x \leq 9 \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x + 256 \int_\Omega \theta^q \|W(Dw + Dg)\|^2 \mathrm{d}x.$$

Then (39) becomes

$$(\nu - 2b\|w\|_{L^\infty(\Omega)}) \int_\Omega \|W(Dw + Dg)\|^2 \theta^q \mathrm{d}x$$
$$\leq \left(337Mn\varepsilon + 2Mn + \frac{Mn}{\varepsilon^{q-1}} + 2a\|u - g\|_{L^\infty(\Omega)}\right) \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x$$
$$+\frac{2nM\gamma^q}{\varepsilon^{q-1}} \sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \, \mathrm{d}x$$
$$+8488nM\varepsilon \int_\Omega \|W(Dw + Dg)\|^2 \theta^q \mathrm{d}x. \tag{40}$$

Finally, choosing $\varepsilon^*$ such that

$$\frac{1}{2}(\nu - 2b\|w\|_{L^\infty(\Omega)}) = 8488nM\varepsilon^*$$

from (7), (9), (40) we get

$$\int_{B(x^0,\sigma)} \|Dw\|^q \, \mathrm{d}x \leq c\sigma^{-q} \int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \, \mathrm{d}x$$
$$+c_1 \int_{B(x^0,2\sigma)} (1 + \|Dg\|)^q \mathrm{d}x$$

with $c = \dfrac{270336n^2M^2\gamma^q}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}$, $c_1 = \dfrac{16a\|u-g\|_{L^\infty(\Omega)}}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}} + \dfrac{16Mn}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}} + \dfrac{8Mn}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}\left(\dfrac{16896nM}{\nu - 2b\|u-g\|_{L^\infty(\Omega)}}\right)^{q-1} + \dfrac{337}{2112}$ and hence the estimate (31).

Now we may prove the interior higher summability of the gradient.

**Theorem 8** *Assume that conditions (4)–(7) are fulfilled. Let* $g \in H^{1,s}(\Omega) \cap L^\infty(\Omega)$, $s > q$, *and let* $w \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, $1 < q < 2$, *be a solution of the strongly elliptic system*

$$\int_\Omega \sum_{i=1}^n \left(a^i(x, w+g, Dw+Dg)|D_i\varphi\right) \mathrm{d}x$$
$$= \int_\Omega (B(x, w+g, Dw+Dg)|\varphi)\,\mathrm{d}x, \qquad \forall \varphi \in H_0^{1,q}(\Omega) \cap L^\infty(\Omega) \tag{41}$$

*then there exists a number* $\tilde{r} > 1$ *such that* $Du \in L^{q\tilde{r}}_{\mathrm{loc}}(\Omega)$ *and* $\forall B(x^0, 2\sigma) \subset \Omega$ *it results*

$$\left(\fint_{B(x^0,\sigma)} \|Dw\|^{q\tilde{r}}\mathrm{d}x\right)^{\frac{1}{\tilde{r}}}$$
$$\leq K \fint_{B(x^0,2\sigma)} \|Dw\|^q \mathrm{d}x + K\left(\fint_{B(x^0,2\sigma)} (1+\|Dg\|)^{q\tilde{r}}\,\mathrm{d}x\right)^{\frac{1}{\tilde{r}}}, \tag{42}$$

*where the constant* $K$ *does not depend on* $\sigma$.

*Proof* First let us consider the case $\frac{n}{n-1} \leq q < 2$. It results $\frac{nq}{n+q} \geq 1$ and then by Poincaré inequality it follows

$$\sigma^{-q}\int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x \leq 16\tilde{c}(n,N)\,\sigma^n \left(\fint_{B(x^0,2\sigma)} \|Dw\|^{\frac{nq}{n+q}}\mathrm{d}x\right)^{\frac{n+q}{n}}. \tag{43}$$

In the other case $1 < q < \frac{n}{n-1}$ we have by Hölder inequality

$$\sigma^{-q}\int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x$$
$$\leq \sigma^{-q}\left(\int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^{q\frac{n}{(n-1)q}}\mathrm{d}x\right)^{\frac{(n-1)q}{n}} \left(\int_{B(x^0,2\sigma)} \mathrm{d}x\right)^{\frac{n(1-q)+q}{n}}$$
$$\leq \omega(n)2^{n(1-q)+q}\sigma^{n(1-q)}\left(\int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^{\frac{n}{n-1}}\mathrm{d}x\right)^{\frac{(n-1)q}{n}}. \tag{44}$$

Applying Poincaré inequality in (44), it results

$$\sigma^{-q}\int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x \leq 4\tilde{\omega}(n,N)\,\sigma^n \left(\fint_{B(x^0,2\sigma)} \|Dw\|\mathrm{d}x\right)^q. \tag{45}$$

Then from (43) and (45), we have

$$\sigma^{-q}\int_{B(x^0,2\sigma)} \|w - w_{B(x^0,2\sigma)}\|^q \mathrm{d}x$$
$$\leq 16\ \max\{\tilde{c}(n,N), \tilde{\omega}(n,N)\}\,\sigma^n \left(\fint_{B(x^0,2\sigma)} \|Dw\|^{\max\{\frac{nq}{n+q},1\}}\mathrm{d}x\right)^{\min\{\frac{n+q}{n},q\}}. \tag{46}$$

If we set

$$U = \|Dw\|^{\max\{\frac{qn}{n+q},1\}}, \quad G = (1 + \|Dg\|)^{\max\{\frac{qn}{n+q},1\}},$$

from “Caccioppoli’s inequality” (31) and (46) it follows

$$\fint_{B(x^0,\sigma)} U^{\min\{\frac{n+q}{n},q\}} \mathrm{d}x \le 16 \max\{\tilde{c}(n,N), \tilde{\omega}(n,N)\} c \left( \fint_{B(x^0,2\sigma)} U \mathrm{d}x \right)^{\min\{\frac{n+q}{n},q\}}$$
$$+ 2^n c_1 \fint_{B(x^0,2\sigma)} G^{\min\{\frac{n+q}{n},q\}} dx.$$

Using Lemma 1, we obtain that there exists a number $r > 1$ such that $Dw \in L^{rq}_{\mathrm{loc}}(\Omega)$. Moreover, taking into account that the constants $c$ and $c_1$ do not depend on $q$, the increment of summability $r$ for $1 < q_0 \le q < 2$ depends only on $q_0$ and does not depend on $q$.

The higher summability up to the boundary and the global higher summability for the gradient may be obtained following the same steps as in the case $q \ge 2$ and then Theorem 2 is completely proved.

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